Final Exam Practice--Problems

Problem

1. Consider the function \( f(x) = 2(x - 1)^2 - 3 \).
   a) Determine the equation of each function.
      i) \(-f(x)\)
      ii) \(f(-x)\)
      iii) \(-f(-x)\)
   b) Graph all four functions from part a) on the same set of axes.
   c) From the graph, determine the pairs of equations that can be represented as translations of each other.
   d) Describe the translation that can be applied to each pair of functions you determined in part c) to generate the same graph.

2. For the base function \( f(x) = x^2 \), a reflection in the \( y \)-axis is followed by a reflection in the \( x \)-axis.
   a) Discuss the effect of each reflection on the base function.
   b) Is there a point that you expect to be invariant during these reflections? If so, state its coordinates.
   c) Graph the function after each reflection.
   d) Use your graph in part c) to verify your answer for part b).
   e) Make a general statement regarding invariant points after reflections in the \( x \)-axis and the \( y \)-axis.

3. The base function \( f(x) = \sqrt{x} \) is reflected in the \( x \)-axis, stretched horizontally by a factor of 2, compressed vertically by a factor of \( \frac{1}{3} \), and translated 3 units to the left and 5 units down.
   a) Write the equation of the transformed function \( g(x) \).
   b) Graph the original function and the transformed function on the same set of axes.
   c) Which transformations must be done first but in any order?
   d) Which transformations must be done last but in any order?

4. Consider the function \( f(x) = -2\sqrt{x + 1} - 4 \).
   a) State the domain and range of the function.
   b) Use this information to determine the domain and range of the inverse of the function.
   c) Determine the inverse of the function.
   d) Graph the function and its inverse on the same set of axes (include the line \( y = x \) to verify the inverse).

5. The graph of \( f(x) = x^2 \) is transformed to the graph of \( g(x) = f(x + 8) + 12 \).
   a) Describe the two translations represented by this transformation.
   b) Determine three points on the base function. Horizontally translate and then vertically translate the points. What are the three resulting image points?

6. Given the function \( f(x) = 3x + 7 \),
   a) determine the inverse of the function
   b) graph the function and its inverse on the same set of axes
7. Consider the function \( f(x) = 2(x - 1)^2 - 3 \).
   a) Determine the square root of the function.
   b) Graph the functions on the same set of axes.
   c) State the domains and ranges of the graphs.
   d) Describe the relationship between the domain and range of \( f(x) \) and the domain and range of \( g(x) \).

8. For \( f(x) = 5\sqrt{-x} \) and \( g(x) = -2\sqrt{6(x + 2)} - 3 \), do the following.
   a) Graph \( f(x) \) and \( g(x) \) on the same set of axes.
   b) Determine the domain and range of each function.
   c) Explain which transformations would need to be applied to the graph of \( f(x) \) to obtain the graph of \( g(x) \).

9. Factor \( 2x^4 - 7x^3 - 41x^2 - 53x - 21 \) fully.

10. Given that \(-2\) is a root of \( x^3 + x = -4x^2 + 6 \), find the other root(s).

11. Solve \( 2x^4 - x^3 = 7x^2 - 9x + 3 \).

12. Determine an equation in expanded form for the polynomial function represented by the graph.

13. Determine an equation in factored form for the polynomial function represented by the graph.

14. The point \((-5, 7)\) is located on the terminal arm of \( \angle A \) in standard position.
   a) Determine the primary trigonometric ratios for \( \angle A \).
   b) Determine the primary trigonometric ratios for \( \angle B \) with the same sine as \( \angle A \), but different signs for the other two primary trigonometric ratios.
   c) Use a calculator to determine the measures of \( \angle A \) and \( \angle B \), to the nearest degree.
15. a) Without using a calculator, determine two angles between 0° and 360° that have a sine ratio of $-\frac{1}{2}$.
   b) Use a calculator and a diagram to verify your answers to part a).

16. Consider ∠A such that $\cos A = \frac{12}{13}$.
   a) In which quadrant(s) is this angle? Explain.
   b) If the sine of the angle is negative, in which quadrant is the angle? Explain.
   c) Sketch a diagram to represent the angle in standard position, given that the condition in part b) is true.
   d) Find the coordinates of a point on the terminal arm of the angle.
   e) Write exact expressions for the other two primary trigonometric ratios for the angle.

17. The point P(−3, −6) lies on the terminal arm of an angle in standard position.
   a) Which primary trigonometric ratios are positive and which are negative?
   b) Which reciprocal trigonometric ratios are positive and which are negative?
   c) Determine exact values for the primary trigonometric ratios.
   d) Determine exact values for the reciprocal trigonometric ratios.

18. Darren cuts a slice from his circular birthday cake, which has a diameter of 30 cm. The slice is in the shape of a sector with arc length 8 cm. What is the measure of the central angle of the slice, in radians, rounded to two decimal places?

19. Sketch the graph of $y = \frac{1}{2} \sin \left( \pi (x - 1) \right) + 3$ for two cycles, where angles are in radians.

20. A windmill has blades that are 20 m in length, and the centre of their circular motion is a point 23 m above the ground. The blades have a frequency of 4 revolutions per minute when in operation.
   a) Use a sinusoidal function to model the height above the ground of the tip of one blade as a function of time.
   b) Graph the function over three complete cycles.
   c) How far above the ground is the tip of the blade after 10 s?

21. Consider the function $f(x) = \frac{1}{2} \sin \left[ 3(x - 30^\circ) \right] + 4$.
   a) Determine the amplitude, the period, the phase shift, and the vertical shift of the function with respect to $y = \sin x$.
   b) What are the minimum and maximum values of the function?
   c) Determine the x-intercepts of the function in the interval $0^\circ \leq x \leq 360^\circ$.
   d) Determine the y-intercept of the function.

22. Prove the identity $1 + \cos \theta = \frac{\sin^2 \theta}{1 - \cos \theta}$.

23. Prove the identity $\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} = \frac{2}{\sin^2 \theta}$.
24. Prove the identity \( \frac{1 + \tan \theta}{1 + \cot \theta} = \frac{1 - \tan \theta}{\cot \theta - 1} \).

25. Prove that the equation \( 2 \csc^2 x = \frac{1}{1 + \cos x} + \frac{1}{1 - \cos x} \) is true for all values of \( x \).

26. Graph each function. Identify the
   i) domain
   ii) range
   iii) intercepts
   iv) intervals of increase/decrease
   v) equation of the asymptote

   a) \( f(x) = \left( \frac{1}{3} \right)^x \)

   b) \( g(x) = -(2)^{-x} \)

27. a) Rewrite the function \( y = 2^{-2x+4} + 6 \) in the form \( y = a(2)^{b(x-h)} + k \).
   
   b) Describe the transformations that must be applied to the graph of \( y = 2^x \) to obtain the graph of the given function.
   
   c) Graph the function.
   
   d) Determine the equation of the function that results after the graph in part c) is reflected in the \( x \)-axis.
   
   e) Graph the function from part d).

28. List the steps and explain the effect of each transformation to graph the function \( y = -3\log\left[-2(x-1)\right] + 4 \).
29. a) Sketch the graph of the function \( f(x) = 3 \log \left[ 2(x - 1) \right] + 4 \). Identify the key features.

b) Identify the domain and range.

c) Identify the equation of the vertical asymptote.

30. A 200-g sample of a radioactive substance is placed in a chamber to be tested. After 3 h, 140 g of the sample remains.

a) Determine the half-life of this substance, to the nearest hundredth of an hour.

b) Graph the amount of the substance remaining as a function of time.
31. Write an equation for the graph of the rational function shown. Explain your reasoning.
32. a) The graph of a function of the form \( f(x) = \frac{ax + b}{cx + d} \) is shown. Determine the key features:
   i) domain and range
   ii) intercepts
   iii) equations of any asymptotes

   b) State a possible equation for this function.

33. Solve.
   a) \( x + 2 = \frac{2}{x + 3} \)
   b) \( \frac{2x}{x - 2} = 2 - \frac{1}{x + 2} \)
   c) \( \frac{2x + 1}{x + 2} = \frac{x + 3}{x + 4} \)
   d) \( \frac{1}{x} + \frac{1}{x^2} = \frac{1}{x^3} \)

34. What is the solution to the rational equation \( \frac{x - 2}{x + 4} + \frac{x + 1}{x + 6} = \frac{11x + 32}{x^2 + 10x + 24} \)?

35. Given the functions \( f(x) = x^2 + 3x + 2 \) and \( g(x) = x + 1 \), determine a simplified equation for \( k(x) = f(g(x)) \).
36. Given the functions \( f(x) = x^2 - 3x - 10 \) and \( g(x) = x^2 - 5x \), graph the function \( h(x) = \frac{f(x)}{g(x)} \). Label all intercepts and identify any asymptotes and/or points of discontinuity.

37. A standard deck of playing cards contains 4 suits (spades, clubs, diamonds, and hearts), each with 13 cards.
   a) How many different 5-card hands are possible?
   b) How many different 5-card hands with only black cards are possible?
   c) How many different 5-card hands are possible containing at least 3 black cards?

38. The chorus of a play has 17 females and 13 males. The director wishes to meet with 6 of them to discuss the upcoming production.
   a) How many selections are possible?
   b) How many selections are possible if the group consists of three females and three males?
   c) One of the male students is named Ajay. How many six-member selections consisting of Ajay, two other males, and three females are possible?

39. Expand using the binomial theorem: \( \left( x^2 - \frac{x}{2} \right)^5 \).

40. Which of the following binomial expansions has the greater coefficient: the 12th term in the expansion of \( (2x - 2)^{13} \) or the 8th term in the expansion of \( (3x - 2)^{10} \)?
Final Exam Practice--Problems
Answer Section

PROBLEM

1. ANS:
   a) i) \( f(x) = -(2(x - 1)^2 - 3) \)
      \[ = -2(x - 1)^2 + 3 \]
   ii) \( f(-x) = 2(-x - 1)^2 - 3 \)
      \[ = 2(-1)^2 (x + 1)^2 - 3 \]
      \[ = 2(x + 1)^2 - 3 \]
   iii) \( -f(-x) = -[2(-x - 1)^2 - 3] \)
      \[ = -2(-1)^2 (x + 1)^2 - 3 \]
      \[ = -2(x + 1)^2 - 3 \]
      \[ = -2(x + 1)^2 + 3 \]
   
   b) 

   c) The graphs of \( f(x) \) and \( f(-x) \) are horizontal translations of each other. This is also true of \( -f(x) \) and \( -f(-x) \).
   d) In both pairs, one curve is a horizontal translation of 2 units left or right of the other.

PTS: 1    DIF: Difficult    OBJ: Section 1.3    NAT: RF4
TOP: Combining Transformations    KEY: reflection | graph | translation | function notation
2. ANS:
a) The reflection in the \( y \)-axis does not change the graph of the function. The reflection in the \( x \)-axis does change the graph, so that the graph of the function points downward.
b) Since the vertex is on the \( x \)-axis and the \( y \)-axis, at \((0, 0)\), this point should not change during the reflections.

c) 

\[
\begin{array}{c}
\begin{pmatrix}
-4 & -2 & 2 & 4 \\
2 & 4 & 5 & 6
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
-4 & -2 & 2 & 4 \\
2 & 4 & 5 & 6
\end{pmatrix}
\end{array}
\]

d) The graph shows that the point \((0, 0)\) is invariant under the reflections.
e) Invariant points after a reflection in the \( x \)-axis are those points of the original function that are on the \( x \)-axis, and invariant points after a reflection in the \( y \)-axis are those points of the original function that are on the \( y \)-axis. The only invariant point after both reflections is the origin.

3. ANS:
a) \( g(x) = \frac{1}{3} \sqrt{\frac{1}{2} (x + 3)} - 5 \)

b) 

c) The reflection, horizontal stretch, and vertical compression must be done first, but can be done in any order.
d) The translations to the left and down must be done last, but can be done in any order.
4. ANS:
   a) domain \( \{ x \geq -1, x \in \mathbb{R} \} \), range \( \{ y \leq -4, y \in \mathbb{R} \} \)
   b) domain \( \{ x \leq -4, x \in \mathbb{R} \} \), range \( \{ y \geq -1, y \in \mathbb{R} \} \)
   c) 
   \[
   y = -2\sqrt{x + 4} - 1
   \]
   
   \[
   x = -2\sqrt{y + 1} - 4
   \]
   
   \[
   x + 4 = -2\sqrt{y + 1}
   \]
   
   \[
   -\frac{(x + 4)}{2} = \sqrt{y + 1}
   \]
   
   \[
   \left(-\left(\frac{x + 4}{2}\right)\right)^2 = y + 1
   \]
   
   \[
   y = \left(\frac{x + 4}{2}\right)^2 - 1
   \]
   
   \[
   f^{-1}(x) = \frac{1}{4} (x + 4)^2 - 1
   \]
   
   d) The graph of \( f(x) \) is shown in blue and the graph of \( f^{-1}(x) \) is shown in red.

5. ANS:
   a) The graph is translated 8 units to the left and 12 units upward.
   b) Answers may vary. Sample answer:
   Start with the points \((0, 0)\), \((1, 1)\), and \((-1, 1)\). Translating the points 8 units to the left results in the coordinates \((-8, 0)\), \((-7, 1)\), and \((-9, 1)\). Translating the points 12 units upward changes the coordinates to \((-8, 12)\), \((-7, 13)\), and \((-9, 13)\).

PTS: 1 DIF: Difficult OBJ: Section 1.4 NAT: RF6
TOP: Inverse of a Relation
KEY: graph | inverse of a function | function notation | domain | range

PTS: 1 DIF: Easy OBJ: Section 1.1 NAT: RF2
TOP: Horizontal and Vertical Translations
KEY: horizontal translation | vertical translation
6. ANS:
   a) \( f(x) = 3x + 7 \)
      \[ y = 3x + 7 \]
      \[ x = 3y + 7 \]
      \[ x - 7 = 3y \]
      \[ y = \frac{x - 7}{3} \]
      \( f^{-1}(x) = \frac{x - 7}{3} \)

   b) [Graph of the function and its inverse]

   PTS: 1  DIF: Easy  OBJ: Section 1.4  NAT: RF6
   TOP: Inverse of a Relation  KEY: inverse of a function | graph
ANS: 

a) \( g(x) = \sqrt{f(x)} \)

\[ g(x) = \sqrt{2(x - 1)^2 - 3} \]

b) 

c) The domain of \( f(x) \) is \( \{ x | x \in \mathbb{R} \} \) and the domain of \( g(x) \) is \( \{ x | x \leq 1 - \sqrt{\frac{3}{2}}, x \geq 1 + \sqrt{\frac{3}{2}}, x \in \mathbb{R} \} \). The range of \( f(x) \) is \( \{ y | y \geq -3, y \in \mathbb{R} \} \) and the range of \( g(x) \) is \( \{ y | y \geq 0, y \in \mathbb{R} \} \).

d) The domain of \( g(x) \) consists of all values of the domain of \( f(x) \) where \( f(x) \geq 0 \). The range of \( g(x) \) consists of the square root of all positive values of the range of \( f(x) \).
8. ANS:

a) The graph of the function is shown. The domain is {x | x ≤ 0, x ∈ R} and the range is {y | y ≥ 0, y ∈ R}.

b) $f(x)$: Domain: $\{x \mid x \leq 0, x \in \mathbb{R}\}$; Range: $\{y \mid y \geq 0, y \in \mathbb{R}\}$

g(x): Domain: $\{x \mid x \geq -2, x \in \mathbb{R}\}$; Range: $\{y \mid y \leq -3, y \in \mathbb{R}\}$

c) The vertical stretch changes from 5 to 2, which means a vertical compression by a factor of $\frac{2}{5}$. There is a horizontal compression by a factor of $\frac{1}{6}$. The graph is reflected in both the x-axis and the y-axis. The graph is translated 2 units left and 3 units down.

PTS: 1  DIF: Difficult  OBJ: Section 2.1  NAT: RF13
TOP: Radical Functions and Transformations  
KEY: graph | transformations | domain | range
9. ANS:
Let \( P(x) = 2x^4 - 7x^3 - 41x^2 - 53x - 21 \).
Test \( x = -1 \) in the factor theorem.
\[
P(x) = 2x^4 - 7x^3 - 41x^2 - 53x - 21
\]
\[
P(-1) = 2(-1)^4 - 7(-1)^3 - 41(-1)^2 - 53(-1) - 21
= 2 + 7 + 41 + 53 + 21
P(-1) = 0
\]
Thus, \( x + 1 \) is a factor. Divide to determine another factor.
\[
\begin{array}{c|cccc}
1 & 2 & -7 & -41 & -53 & -21 \\
- & & 2 & -9 & -32 & -21 \\
\hline
2 & -9 & -32 & -21 & 0 \\
\end{array}
\]
Thus,
\[
P(x) = (x + 1)(2x^3 - 9x^2 - 32x - 21)
\]
Now factor the cubic. Test \( x = -1 \) in the factor theorem. Let \( Q(x) = 2x^3 - 9x^2 - 32x - 21 \).
\[
Q(x) = 2x^3 - 9x^2 - 32x - 21
\]
\[
Q(-1) = 2(-1)^3 - 9(-1)^2 - 32(-1) - 21
= -2 + 9 + 32 + 21
Q(-1) = 0
\]
Thus, \( x + 1 \) is a factor. Divide to determine another factor.
\[
\begin{array}{c|cccc}
1 & 2 & -9 & -32 & -21 \\
- & & 2 & -11 & -21 \\
\hline
2 & -11 & -21 & 0 \\
\end{array}
\]
Thus,
\[
P(x) = (x + 1)(x + 1)(2x^2 - 11x - 21)
= (x + 1)^2(2x^2 - 14x + 3x - 21)
= (x + 1)^2[2x(x - 7) + 3(x - 7)]
= (x + 1)^2(x - 7)(2x + 3)
\]

PTS: 1  
DIF: Average  
OBJ: Section 3.3  
NAT: RF11  
TOP: The Factor Theorem  
KEY: factor theorem | integral zero theorem | factor
10. ANS:
Rewrite in the form $P(x) = 0$.

$x^3 + 4x^2 + x - 6 = 0$

Since $-2$ is a root, $x + 2$ is a factor of $P(x)$. Divide to find the other factor.

\[
\begin{array}{c|cccc}
2 & 1 & 4 & 1 & -6 \\
\hline 
 & 2 & 4 & -6 \\
\end{array}
\]

$x^3 + 4x^2 + x - 6 = 0$

$(x + 2)(x^2 + 2x - 3) = 0$

$(x + 2)(x + 3)(x - 1) = 0$

$x = -2, -3, 1$

Thus, the other roots are $-3$ and $1$.

PTS: 1 DIF: Average OBJ: Section 3.3 NAT: RF11
TOP: The Remainder Theorem KEY: factor theorem | root
11. ANS:
Rewrite in the form \( P(x) = 0 \).
\[ 2x^4 - x^3 - 7x^2 + 9x - 3 = 0 \]
Try \( x = 1 \) in the factor theorem.
\[ P(x) = 2x^4 - x^3 - 7x^2 + 9x - 3 \]
\[ P(1) = 2(1)^4 - 1^3 - 7(1)^2 + 9(1) - 3 \]
\[ = 2 - 1 - 7 + 9 - 3 \]
\[ P(1) = 0 \]
Since \( x = 1 \) is a zero of \( P(x) \), \( x - 1 \) is a factor. Divide to determine another factor.
\[
\begin{array}{c|cccc}
-1 & 2 & -1 & -7 & 9 & -3 \\
- & 2 & -1 & 6 & -3 \\
\hline
 & 2 & 1 & -6 & 3 & 0
\end{array}
\]
\[ 2x^4 - x^3 - 7x^2 + 9x - 3 = 0 \]
\[ (x - 1)(2x^3 + x^2 - 6x + 3) = 0 \]
Now factor the cubic. Try \( x = 1 \) in the factor theorem. Let \( Q(x) = 2x^3 + x^2 - 6x + 3 \).
\[ Q(x) = 2x^3 + x^2 - 6x + 3 \]
\[ Q(1) = 2(1)^3 + (1)^2 - 6(1) + 3 \]
\[ = 2 + 1 - 6 + 3 \]
\[ Q(1) = 0 \]
Thus, \( x - 1 \) is a factor of \( Q(x) \). Use division to find another factor.
\[
\begin{array}{c|cccc}
-1 & 2 & 1 & -6 & 3 \\
- & -2 & -3 & 3 \\
\hline
 & 2 & 3 & -3 & 0
\end{array}
\]
\[ 2x^4 - x^3 - 7x^2 + 9x - 3 = 0 \]
\[ (x - 1)(2x^3 + x^2 - 6x + 3) = 0 \]
\[ (x - 1)(2x^2 + 3x - 3) = 0 \]
Thus, \( x = 1 \) or \( 2x^2 + 3x - 3 = 0 \). Use the quadratic formula to find the other roots.
\[ x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-3)}}{2(2)} \]
\[ = \frac{-3 \pm \sqrt{33}}{4} \]
The solution is \( x = 1 \), \( x = \frac{-3 + \sqrt{33}}{4} \), or \( x = \frac{-3 - \sqrt{33}}{4} \).
12. ANS:
The graph has a single zero at \( x = 0 \) and a double zero at \( x = 2 \). The graph also passes through the point \((3, 6)\). Thus, the graph is of the form \( y = ax(x - 2)^2 \). Substitute the point \((3, 6)\) to find \( a \).

\[
y = ax(x - 2)^2
\]

\[
6 = a(3)(3 - 2)^2
\]

\[
6 = 3a
\]

\[
a = 2
\]

Thus,

\[
y = 2x(x - 2)^2
\]

\[
= 2x(x^2 - 4x + 4)
\]

\[
y = 2x^3 - 8x^2 + 8x
\]

PTS: 1  DIF: Average  OBJ: Section 3.4  NAT: RF12
TOP: Equations and Graphs of Polynomial Functions  KEY: polynomial equation | graph | zeros

13. ANS:
There are single zeros at \( x = -1 \) and \( x = 1 \), and a double zero at \( x = 2 \). The graph passes through the point \((0, 4)\). Thus, the equation is of the form \( y = a(x + 1)(x - 1)(x - 2)^2 \). Substitute the point \((0, 4)\) to find \( a \).

\[
y = a(x + 1)(x - 1)(x - 2)^2
\]

\[
4 = a(0 + 1)(0 - 1)(0 - 2)^2
\]

\[
4 = -4a
\]

\[
a = -1
\]

Thus,

\[
y = -(x + 1)(x - 1)(x - 2)^2
\]

PTS: 1  DIF: Average  OBJ: Section 3.4  NAT: RF12
TOP: Equations and Graphs of Polynomial Functions  KEY: graph | polynomial equation | zeros | y-intercept
14. ANS:
\( \angle A \) is in quadrant II. Therefore, only the sine ratio will be positive.
Use the Pythagorean theorem.
\[
r^2 = x^2 + y^2
\]
\[
= (-5)^2 + 7^2
\]
\[
= 25 + 49
\]
\[
= 74
\]
\[
r = \sqrt{74}
\]
Therefore, \( \sin A = \frac{7}{\sqrt{74}} \), \( \cos A = -\frac{5}{\sqrt{74}} \), and \( \tan A = -\frac{7}{5} \).

b) The quadrant in which the sine ratio is still positive, but the cosine and tangent ratios change from negative to positive, is quadrant I. In this quadrant, all three primary trigonometric ratios are positive.
\( \sin B = \frac{7}{\sqrt{74}} \), \( \cos B = \frac{5}{\sqrt{74}} \), and \( \tan B = \frac{7}{5} \).
e) Use the fact that \( \angle B \) is the reference angle for \( \angle A \).
\( \angle B = \sin^{-1} \left( \frac{7}{\sqrt{74}} \right) \)
\( \angle B \approx 54^\circ \)
\( \angle A = 180^\circ - 54^\circ \)
\( = 126^\circ \)
15. ANS:

a) Since \( \sin 30^\circ = \frac{1}{2} \), the reference angle is \( 30^\circ \). The sine ratio is negative in quadrants III and IV. Look for reflections of the \( 30^\circ \) angle in these quadrants.

- Quadrant III: \( 180^\circ + 30^\circ = 210^\circ \)
- Quadrant IV: \( 360^\circ - 30^\circ = 330^\circ \)

b) Using a calculator, \( \sin 210^\circ = -\frac{1}{2} \) and \( \sin 330^\circ = -\frac{1}{2} \).
16. ANS:
   a) Since the cosine ratio is positive, the angle is in quadrant I or IV.
   b) If the sine ratio is negative, the angle is in quadrant IV.
   c) Use the Pythagorean theorem.
      
      \[ r^2 = x^2 + y^2 \]
      
      \[ 13^2 = 12^2 + y^2 \]
      
      \[ y^2 = 169 - 144 \]
      
      \[ y^2 = 25 \]
      
      \[ y = \pm 5 \]
      
      Therefore, a point on the terminal arm is (12, –5).

   d) Use the Pythagorean theorem.
      
      \[ r^2 = x^2 + y^2 \]
      
      \[ 13^2 = 12^2 + y^2 \]
      
      \[ y^2 = 169 - 144 \]
      
      \[ y^2 = 25 \]
      
      \[ y = \pm 5 \]
      
      Therefore, a point on the terminal arm is (12, –5).

   e) \[ \sin A = -\frac{5}{13}, \tan A = -\frac{5}{12} \]

PTS: 1  DIF: Average  OBJ: Section 4.2 | Section 4.3
NAT: T2 | T3  TOP: Unit Circle | Trigonometric Ratios
KEY: trigonometric ratios | unit circle | reference angle
17. **ANS:**

a) Since the point on the terminal arm lies in quadrant III, the tangent ratio is positive, and the sine and cosine ratios are negative.

b) The cotangent ratio is positive, and the cosecant and secant ratios are negative.

c) \( r^2 = x^2 + y^2 \)

\[
= (-3)^2 + (-6)^2 \\
= 9 + 36 \\
= 45
\]

\( r = \sqrt{45} \)

\[
= 3\sqrt{5}
\]

\( \sin P = \frac{y}{r} \)

\[
= \frac{-6}{3\sqrt{5}}
\]

\( \cos P = \frac{x}{r} \)

\[
= \frac{-3}{3\sqrt{5}}
\]

\( \tan P = \frac{y}{x} \)

\[
= \frac{-6}{-3}
\]

\[
= 2
\]

**d) Using the answers in part c), take the reciprocal of each primary trigonometric ratio to write the reciprocal trigonometric ratios.**

\( \csc P = \frac{1}{\frac{3\sqrt{5}}{2}} \), \( \sec P = -\frac{1}{\sqrt{5}} \), and \( \cot P = \frac{1}{2} \)

**PTS: 1  DIF: Average  OBJ: Section 4.2 | Section 4.3  
NAT: T2 | T3  TOP: Unit Circle | Trigonometric Ratios  
KEY: primary trigonometric ratios | reciprocal trigonometric ratios | point on terminal arm**

18. **ANS:**

Since the diameter is 30 cm, the radius is 15 cm.

\( a = r\theta \)

\[ 8 = 15\theta \]

\[ \theta = \frac{8}{15} \]

\[ \approx 0.53 \]

The central angle measures 0.53 rad.

**PTS: 1  DIF: Easy  OBJ: Section 4.1  NAT: T1  
TOP: Angles and Angle Measure  
KEY: central angle | arc length**
19. **ANS:**
   The domain of students’ graphs may vary. Sample graph:

![Sample graph](image)

**PTS:** 1  **DIF:** Average  **OBJ:** Section 5.2  **NAT:** T4  
**TOP:** Transformations of Sinusoidal Functions  
**KEY:** sinusoidal function | graph | transformations
20. ANS:
Solutions may vary. Sample solution:

a) The amplitude is 20 m, and the vertical displacement is 23 m. The frequency of the blades is 4 revolutions per minute, so the period is 0.25 min or 15 s. Thus, \( b = \frac{2\pi}{15} \), and the sinusoidal function is

\[
h = 20 \sin \left( \frac{2\pi}{15} t \right) + 23.\]

b) [Graph of sinusoidal function]

e) Substitute \( t = 10 \) into the equation.

\[
h = 20 \sin \left( \frac{2\pi}{15} (10) \right) + 23
\]

\[
= 20 \sin \left( \frac{2\pi (10)}{15} \right) + 23
\]

\[
\approx 5.7
\]

The tip of the blade is approximately 5.7 m above the ground at \( t = 10 \) s.

PTS: 1 \hspace{1cm} DIF: Average \hspace{1cm} OBJ: Section 5.4 \hspace{1cm} NAT: T4
TOP: Equations and Graphs of Trigonometric Functions \hspace{1cm} KEY: model | sinusoidal function | graph
21. ANS: 

a) amplitude: \( \frac{1}{2} \); period: \( \frac{360^\circ}{3} \) or \( 120^\circ \); phase shift: \( 30^\circ \) to the right; vertical shift: 4 units up 

b) minimum: \( -\frac{1}{2} + 4 \) or \( \frac{7}{2} \), maximum: \( \frac{1}{2} + 4 \) or \( \frac{9}{2} \) 

c) The function does not cross the \( x \)-axis, so there are no \( x \)-intercepts. 

d) Substitute \( x = 0 \) in the function: 

\[
f(x) = \frac{1}{2} \sin[3(x - 30^\circ)] + 4
\]

\[
f(0) = \frac{1}{2} \sin[3(0 - 30^\circ)] + 4
\]

\[
= \frac{1}{2} \sin(-90^\circ) + 4
\]

\[
= \frac{1}{2} + 4
\]

\[
= \frac{7}{2}
\]

The \( y \)-intercept is \( \frac{7}{2} \). 

PTS: 1 DIF: Average OBJ: Section 5.2 NAT: T4 TOP: Transformations of Sinusoidal Functions KEY: amplitude | period | shifts | transformations | sine function

22. ANS: 

L.S. = \( 1 + \cos \theta \) 

R.S. = \( \frac{\sin^2 \theta}{1 - \cos \theta} \)

\[
= \frac{1 - \cos^2 \theta}{1 - \cos \theta}
\]

\[
= \frac{(1 - \cos \theta)(1 + \cos \theta)}{1 - \cos \theta}
\]

\[
= 1 + \cos \theta
\]

L.S. = R.S. 

Therefore, \( 1 + \cos \theta = \frac{\sin^2 \theta}{1 - \cos \theta} \). 

PTS: 1 DIF: Average OBJ: Section 6.1 | Section 6.3 NAT: T6 TOP: Reciprocal, Quotient, and Pythagorean Identities | Proving Identities KEY: Pythagorean identities | proof
23. ANS:

\[
\text{L.S.} = \frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} \quad \text{R.S.} = \frac{2}{\sin^2 \theta}
\]

\[
= \frac{1 - \cos \theta + 1 + \cos \theta}{(1 + \cos \theta)(1 - \cos \theta)}
\]

\[
= \frac{2}{1 - \cos^2 \theta}
\]

\[
= \frac{2}{\sin^2 \theta}
\]

\[
\text{L.S.} = \text{R.S.}
\]

PTS: 1 DIF: Average OBJ: Section 6.1 | Section 6.3
NAT: T6 TOP: Reciprocal, Quotient, and Pythagorean Identities | Proving Identities
KEY: Pythagorean identities | proof

24. ANS:

\[
\text{L.S.} = \frac{1 + \tan \theta}{1 + \cot \theta} \quad \text{R.S.} = \frac{1 - \tan \theta}{\cot \theta - 1}
\]

\[
= \frac{1 + \frac{\sin \theta}{\cos \theta}}{1 + \frac{\cos \theta}{\sin \theta}}
\]

\[
= \frac{\cos \theta + \sin \theta}{\sin \theta + \cos \theta}
\]

\[
= \left( \frac{\cos \theta + \sin \theta}{\cos \theta} \right) \left( \frac{\sin \theta}{\cos \theta + \sin \theta} \right)
\]

\[
= \frac{\sin \theta}{\cos \theta}
\]

\[
\text{R.S.} = \frac{1 - \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\sin \theta} - 1}
\]

\[
= \frac{\cos \theta - \sin \theta}{\cos \theta}
\]

\[
= \left( \frac{\cos \theta - \sin \theta}{\cos \theta} \right) \left( \frac{\sin \theta}{\cos \theta - \sin \theta} \right)
\]

\[
= \frac{\sin \theta}{\cos \theta}
\]

\[
\text{L.S.} = \text{R.S.}
\]

PTS: 1 DIF: Average OBJ: Section 6.1 | Section 6.3
NAT: T6 TOP: Reciprocal, Quotient, and Pythagorean Identities | Proving Identities
KEY: quotient identities | proof
ANS: 

L.S. = 2\( \csc^2 x \) 

\[
R.S. = \frac{1}{1 + \cos x} + \frac{1}{1 - \cos x}
\]

\[
= \frac{(1 - \cos x) + (1 + \cos x)}{(1 + \cos x)(1 - \cos x)}
\]

\[
= \frac{2}{1 - \cos^2 x}
\]

\[
= \frac{2}{\sin^2 x}
\]

\[
= 2 \csc^2 x
\]

L.S. = R.S.

This proves that the expression is valid for all values of \( x \).
26. ANS:

a) 

\[ y = \begin{cases} 
\text{domain } \{ x \mid x \in \mathbb{R} \} \\
\text{range } \{ y \mid y > 0, y \in \mathbb{R} \} \\
\text{no } x\text{-intercept; } y\text{-intercept is } 1 \\
\text{the function is always decreasing} \\
y = 0 
\end{cases} \]

b) 

\[ y = \begin{cases} 
\text{domain } \{ x \mid x \in \mathbb{R} \} \\
\text{range } \{ y \mid y < 0, y \in \mathbb{R} \} \\
\text{no } x\text{-intercept; } y\text{-intercept is } -1 \\
\text{the function is always increasing} \\
y = 0 
\end{cases} \]
27. ANS:

a) \( y = 2^{-2(x-2)} + 6 \)

b) Reflect in the y-axis, compress horizontally by a factor of \( \frac{1}{2} \), and translate 2 units to the right and 6 units up.

e) [Graph of a transformed exponential function]

d) \( y = -2^{-2(x-2)} - 6 \)

e) [Graph of a transformed exponential function]

PTS: 1  DIF: Average  OBJ: Section 7.2  NAT: RF9
TOP: Exponential Functions  KEY: graph | transformations of exponential functions

28. ANS:

1. vertically stretched by a factor of 3
2. horizontally compressed by a factor of \( \frac{1}{2} \),
3. reflected in both the x- and y-axes
4. translated 1 unit right and 4 units up

PTS: 1  DIF: Average  OBJ: Section 8.2  NAT: RF8
TOP: Transformations of Logarithmic Functions  KEY: transformations | logarithmic functions
29. ANS:

a) 

\[ \begin{array}{c}
\text{domain: } \{x|x > 1, x \in \mathbb{R}\}; \\
\text{range: } \{y|y \in \mathbb{R}\} \\
\text{c) } x = 1
\end{array} \]

b) domain: \( \{x|x > 1, x \in \mathbb{R}\} \); range: \( \{y|y \in \mathbb{R}\} \)

c) \( x = 1 \)

PTS: 1 DIF: Difficult OBJ: Section 8.2 NAT: RF8
TOP: Transformations of Logarithmic Functions
KEY: transformation | domain | range | asymptote
30. ANS:
   a) Let \( h \) represent the half-life of the substance.

   \[
   140 = 200 \left( \frac{1}{2} \right)^{\frac{3}{h}}
   \]

   \[
   0.7 = 0.5^\frac{3}{h}
   \]

   \[
   \log 0.7 = \frac{3}{h} \log 0.5
   \]

   \[
   h = \frac{3 \log 0.5}{\log 0.7}
   \]

   \[
   h \approx 5.83
   \]

   The half-life is 5.83 h.

   b) Graph \( y = 200 \left( \frac{1}{2} \right)^{\frac{x}{5.83}} \).

PTS: 1     DIF: Difficult     OBJ: Section 8.4     NAT: RF10
TOP: Logarithmic and Exponential Equations
KEY: exponential equation | logarithmic equation
31. **ANS:**
   Answers may vary. Sample answer:
   \[ f(x) = \frac{1}{x^2 - 4} \]
   Any function of the form \( f(x) = \frac{k}{x^2 - 4}, \quad k > 0 \), is a reasonable candidate since it is difficult to tell from the graph how stretched the function is.

32. **ANS:**
   a) i) \( \{ x | x \neq -2, \quad x \in \mathbb{R} \}, \{ y | y \neq 2, \quad y \in \mathbb{R} \} \)
   ii) x-intercept: –3, y-intercept: 3
   iii) \( x = -2, \quad y = 2 \)
   b) Answers may vary. Sample answer:
   \[ f(x) = \frac{2x + 6}{x + 2} \]
   (Since there is a vertical asymptote at \( x = -2 \), the denominator must be a multiple of \( x + 2 \). Since there is a horizontal asymptote at \( y = 2 \), the ratio \( \frac{a}{c} = 2 \). If you pick \( c = 1 \), then \( a = 2 \). To ensure an x-intercept of –3 and a y-intercept of 3, \( b = 6 \).

33. **ANS:**
   a) \( x = -4 \) or \( x = -1 \)
   b) \( x = \frac{6}{5} \)
   c) \( x = -2 \pm \sqrt{6} \)
   d) No solution.

**PTS: 1** **DIF: Average** **OBJ: Section 9.2** **NAT: RF14**
**TOP: Analysing Rational Functions** **KEY: reciprocal of quadratic function**

**PTS: 1** **DIF: Average** **OBJ: Section 9.2** **NAT: RF14**
**TOP: Analysing Rational Functions** **KEY: linear expressions in numerator and denominator | graph | key features**

**PTS: 1** **DIF: Average** **OBJ: Section 9.3** **NAT: RF14**
**TOP: Connecting Graphs and Rational Equations** **KEY: rational equation**
34. ANS: 
\[
\frac{x - 2}{x + 4} + \frac{x + 1}{x + 6} = \frac{11x + 32}{x^2 + 10x + 24} \tag{1}
\]
\[
\frac{x - 2}{x + 4} + \frac{x + 1}{x + 6} = \frac{11x + 32}{(x + 4)(x + 6)} \tag{2}
\]
\[
(x + 4)(x + 6) \left( \frac{x - 2}{x + 4} + \frac{x + 1}{x + 6} \right) = \left( \frac{11x + 32}{(x + 4)(x + 6)} \right) (x + 4)(x + 6)
\]
\[
(x - 2)(x + 6) + (x + 4)(x + 1) = 11x + 32
\]
\[
x^2 + 4x - 12 + x^2 + 5x + 4 = 11x + 32
\]
\[
2x^2 - 2x - 40 = 0
\]
\[
x^2 - x - 20 = 0
\]
\[
(x - 5)(x + 4) = 0
\]
\[
x = 5, \; x = -4
\]
Check:
Let \(x = 5\).
\[
\text{L.S.} = \frac{5 - 2}{5 + 4} + \frac{5 + 1}{5 + 6} = \frac{3}{9} + \frac{6}{11} = \frac{33}{99} + \frac{54}{99} = \frac{87}{99}
\]
\[
\text{R.S.} = \frac{11(5) + 32}{5^2 + 10(5) + 24} = \frac{87}{99}
\]
\[
\text{L.S.} = \text{R.S.}
\]
The value of \(x\) cannot be \(-4\), because \(x = -4\) is an inadmissible value.
Therefore, the solution is \(x = 5\).

PTS: 1 DIF: Difficult OBJ: Section 9.3 NAT: RF14 TOP: Connecting Graphs and Rational Equations KEY: quadratic denominator | extraneous solution

35. ANS: 
\[
k(x) = (x + 1)^2 + 3(x + 1) + 2
\]
\[
= x^2 + 2x + 1 + 3x + 3 + 2
\]
\[
= x^2 + 5x + 6
\]

PTS: 1 DIF: Average OBJ: Section 10.3 NAT: RF1 TOP: Composite Functions KEY: composite functions | evaluate
36. ANS:

\[ C_5 = 2 \, 598 \, 960 \]

b) There are 26 black cards (spades and clubs), so there are \( 26 \, C_5 \) or 65 780 possible hands with only black cards.

c) There are 3 possible situations:

Case 1: all 5 cards are black:
\[ 26 \, C_5 = 65 \, 780 \]

Case 2: 4 black cards, 1 red card
\[ P(\text{black}) \times P(\text{red}) = (26 \, C_4) \times (26 \, C_1) \]
\[ = 388 \, 700 \]

Case 3: 3 black cards, 2 red card
\[ P(\text{black}) \times P(\text{red}) = (26 \, C_3) \times (26 \, C_2) \]
\[ = 845 \, 000 \]

The total number of different 5-card hands containing at least 3 black cards is
\[ 65 \, 780 + 388 \, 700 + 845 \, 000 = 1 \, 299 \, 480. \]

37. ANS:

\[ 52 \, C_5 = 2 \, 598 \, 960 \]

b) There are 26 black cards (spades and clubs), so there are \( 26 \, C_5 \) or 65 780 possible hands with only black cards.

c) There are 3 possible situations:

Case 1: all 5 cards are black:
\[ 26 \, C_5 = 65 \, 780 \]

Case 2: 4 black cards, 1 red card
\[ P(\text{black}) \times P(\text{red}) = (26 \, C_4) \times (26 \, C_1) \]
\[ = 388 \, 700 \]

Case 3: 3 black cards, 2 red card
\[ P(\text{black}) \times P(\text{red}) = (26 \, C_3) \times (26 \, C_2) \]
\[ = 845 \, 000 \]

The total number of different 5-card hands containing at least 3 black cards is
\[ 65 \, 780 + 388 \, 700 + 845 \, 000 = 1 \, 299 \, 480. \]
38. ANS:
   a) \( \binom{30}{6} = \frac{30!}{(30-6)!6!} \)
      \[ = \frac{30!}{24!6!} \]
      \[ = 593775 \]
   There are 593 775 possible ways of selecting 6 members of the chorus.
   b) \( (\binom{17}{3})(\binom{13}{3}) = \left( \frac{17!}{(17-3)!3!} \right) \left( \frac{13!}{(13-3)!3!} \right) \)
      \[ = 680(286) \]
      \[ = 194480 \]
   There are 194 480 possible ways of selecting three females and three males.
   c) There is only one way of selecting Ajay. This leaves 12 males left to choose the other two males.
      \( (1)(\binom{12}{2})(\binom{17}{3}) = 66(680) \)
      \[ = 44880 \]
   There are 44 880 possible ways to select the group that includes Ajay.

39. ANS:
   \[ \left( x^2 - \frac{x}{2} \right)^5 \]
   \[ = 5 C_0 (x^2)^5 \left( \frac{x}{2} \right)^0 + 5 C_1 (x^2)^4 \left( \frac{x}{2} \right)^1 + 5 C_3 (x^2)^2 \left( \frac{x}{2} \right)^3 + 5 C_4 (x^2)^1 \left( \frac{x}{2} \right)^4 + 5 C_5 (x^2)^0 \left( \frac{x}{2} \right)^5 \]
   \[ = x^{10} - \frac{5}{2} x^9 + \frac{5}{2} x^8 - \frac{5}{4} x^7 + \frac{5}{16} x^6 - \frac{x^5}{32} \]

40. ANS:
   12th term in expansion of \((2x - 2)^{13}\):
      \( \binom{13}{11}(2x)^{13-11}(-2)^{11} = 78(2x)^2(-2048) \)
      \[ = -638 976x^2 \]
   8th term in expansion of \((3x - 2)^{10}\):
      \( \binom{10}{7}(3x)^{10-7}(-2)^7 = 120(3x)^3(-128) \)
      \[ = -414 720x^3 \]
Since \(-414 720 > -638 976\), the 8th term in the expansion of \((3x - 2)^{10}\) has the greater coefficient.